Counting Modular Matrices With Specified Maximum Norm

Aloys Krieg*
Lehrstuhl A für Mathematik
RWTH Aachen
D-52056 Aachen, Germany

Submitted by Richard A. Brualdi

ABSTRACT

Let $N(x)$ denote the number of matrices in $SL_2(\mathbb{Z})$ with maximum norm $\leqslant x$. An explicit formula for $N(x)$ is derived, which is basically the partial sum of Euler's totient function $\varphi(n)$. This leads to the asymptotic result $N(x) \sim (96/\pi^2)x^2$.

A. Terras [7, p. 267] and M. Newman [6] gave an asymptotic result on the number of matrices M in the modular group $SL_2(\mathbb{Z})$, whose euclidean norm $\sqrt{\operatorname{tr}({}^tMM)}$ does not exceed x. As an application of the hyperbolic lattice point theorem this was generalized to congruence subgroups of the modular group as well as certain groups over Clifford numbers acting on the k-dimensional hyperbolic space by J. Elstrodt, F. Grunewald, and J. Mennicke [2, Theorem 0.6 and Corollary 4.3]. Another generalization to matrices in $SL_n(\mathbb{Z})$ is due to D. Grenier [4], who applied a noneuclidean version of the Poisson summation formula (cf. [8]). This result was used in order to deal with the analogous problem having the maximum norm in place of the euclidean norm. Grenier [4] showed that the corresponding number of matrices $M \in SL_n(\mathbb{Z})$ with maximum norm $\leqslant x$ is $O(x^{n^2-n+\varepsilon})$ as $x \to \infty$. A related problem, to determine the probability that r integral matrices have relatively prime determinants, was dealt with by J. Hafner, K. McCurley, and P. Sarnak [5].

*e-mail: `krieg@rwth-aachen.de`

LINEAR ALGEBRA AND ITS APPLICATIONS 196:273–278 (1994)

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

In this note we find the number $N(x)$ of matrices M in the modular group $\mathrm{SL}_2(\mathbb{Z})$ whose maximum norm does not exceed x. We derive an explicit formula in a completely elementary way. $N(x)$ is basically the partial sum of Euler's totient function $\varphi(n)$ (see the Theorem). Well-known estimates of Euler's totient function finally lead to the asymptotic behavior, namely $N(x) \sim (96/\pi^2)x^2$.

It is remarkable that our proof only involves elementary number theory, whereas Newman's proof [6] is technical and the proofs in [2] and [7] require a strong background in hyperbolic geometry.

We always write a real 2×2 matrix M in the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define the maximum norm of M by

$$\|M\| := \max\{|a|, |b|, |c|, |d|\}.$$

Given a real number x, set

$$N(x) := \#\{M \in \mathrm{SL}_2(\mathbb{Z}) : \|M\| \leqslant x\}.$$

Hence $N(x)$ counts the solutions

$$ad - bc = 1, \qquad |a|, |b|, |c|, |d| \leqslant x, \quad a, b, c, d \in \mathbb{Z}.$$

First we need the elementary

LEMMA. *Let a, b be relatively prime integers satisfying $a > b \geqslant 1$. Then there exist $c, d \in \mathbb{Z}$ such that*

$$ad - bc = 1 \quad \textit{and} \quad 0 < d \leqslant b, \quad 0 < c < a.$$

Proof. Choose $u, v \in \mathbb{Z}$ such that $au - bv = 1$. Then determine $n \in \mathbb{Z}$ with $0 < v + na =: c < a$. Hence $d := u + nb = (bc + 1)/a$ satisfies $0 < d \leqslant b$. ∎

Now set

$$P(n) := \{M \in \mathrm{SL}_2(\mathbb{Z}) : \|M\| = n\}.$$

Then an explicit calculation yields

$$\#P(1) = 20. \tag{$*$}$$

If φ denotes Euler's totient function, we get the following

PROPOSITION. *Given $n > 1$, one has*

$$\#P(n) = 32\varphi(n).$$

Proof. Given $M \in P(n)$, exactly one entry of M is $\pm n$, because $n > 1$. We may multiply by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

from the left and right hand sides, when necessary, in order to obtain $a = \pm n$. Next multiply by

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and conjugate by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

if necessary, and assume $a = n$ as well as $0 < b < n$. Hence we have

$$\#P(n) = 16\#Q(n), \qquad Q(n) := \left\{M = \begin{pmatrix} n & b \\ c & d \end{pmatrix} \in P(n) : 0 < b < n\right\}.$$

If $M \in Q(n)$, then b and n are relatively prime. Given $0 < b < n$ such that b and n are relatively prime, there exists

$$M = \begin{pmatrix} n & b \\ c & d \end{pmatrix} \in Q(n)$$

due to the lemma. Any two matrices in $\mathrm{SL}_2(\mathbb{Z})$ with (n, b) as their first rows differ by a factor

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

on the left hand side. According to the lemma, only the cases $m = 0, -1$ occur for matrices in $Q(n)$. Hence we have

$$\#Q(n) = 2\varphi(n). \qquad \blacksquare$$

Using $(*)$ and the proposition, we obtain our main result.

THEOREM. *Given $x \geqslant 1$, one has*

$$N(x) = 32 \sum_{n \leqslant x} \varphi(n) - 12.$$

The well-known average order of Euler's totient function (cf. [1, Theorem 3.7]) yields the final

COROLLARY.

$$N(x) = \frac{96}{\pi^2}x^2 + O(x \log x) \qquad as \quad x \to \infty;$$

in particular,

$$N(x) \sim \frac{16}{\zeta(2)}x^2.$$

REMARKS.

(a) In the case of the euclidean norm the asymptotic is $6x^2$ according to [6] or [7, p. 267]. A comparison with the asymptotics for the euclidean lattice point theorem in $\mathbb{R}^4$ (cf. [3, p. 36]) yields

$$\frac{\#\left\{M \in \mathrm{SL}_2(\mathbb{Z}) : \sqrt{\mathrm{tr}({}^tMM)} \leqslant x\right\}}{\#\left\{M \in \mathrm{Mat}_2(\mathbb{Z}) : \sqrt{\mathrm{tr}({}^tMM)} \leqslant x\right\}} \sim \frac{12}{\pi^2 x^2}.$$

But in the case of the maximum norm we obtain

$$\frac{\#\{M \in \mathrm{SL}_2(\mathbb{Z}) : \|M\| \leqslant x\}}{\#\{M \in \mathrm{Mat}_2(\mathbb{Z}) : \|M\| \leqslant x\}} \sim \frac{6}{\pi^2 x^2}.$$

(b) Just as in [4], we get

$$\#\{M \in \mathrm{Mat}_2(\mathbb{Z}) : \det M = m,\ \|M\| \leqslant x\} = O(x^2)$$

for fixed $m \in \mathbb{Z}$, $m \neq 0$. Each $M \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det M = m$ possesses a unique representation

$$M = AB, \qquad A \in \mathrm{SL}_2(\mathbb{Z}), \quad B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad 0 \leqslant b < d,\ ad = m.$$

Now $\|M\| \leqslant x$ yields $\|A\| \leqslant 2x$; hence

$$\#\{M \in \mathrm{Mat}_2(\mathbb{Z}) : \det M = m,\ \|M\| \leqslant x\} \leqslant \sigma_1(|m|) N(2x).$$

(c) If $\mathscr{G}$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index μ, one should expect (as the referee suggested) that

$$\#\{M \in \mathscr{G} : \|M\| \leqslant x\} \sim \frac{96}{\mu\pi^2} x^2,$$

similarly to the case of the euclidean norm (cf. [2]). This can be proved by the method above if $\mathscr{G}$ is the principal congruence subgroup of level 2. But the general case does not seem to be accessible in this way.

REFERENCES

1 T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.
2 J. Elstrodt, F. Grunewald, and J. Mennicke, Arithmetic applications of the hyperbolic lattice point theorem, *Proc. London Math. Soc.* (3) 57:239–283 (1988).
3 F. Fricker, *Einführung in die Gitterpunktlehre*, Birkhäuser, Boston, 1982.
4 D. Grenier, A non-euclidean circle problem, Preprint, Johns Hopkins Univ., Baltimore, 1992.

5 J. Hafner, K. McCurley, and P. Sarnak, Relatively Prime Values of Polynomials, IBM Research Report RJ 7583 (70760), 13 July 1990.
6 M. Newman, Counting modular matrices with specified euclidean norm, *J. Combin. Theory Ser. A* 47:145–149 (1988).
7 A. Terras, *Harmonic analysis on symmetric spaces and applications I*, Springer-Verlag, New York, 1985.
8 A. Terras, *Harmonic analysis on symmetric spaces and applications II*, Springer-Verlag, New York, 1988.

Received 23 November 1992; final manuscript accepted 13 January 1993